## Background

Calculus II

This is a continuation of Calculus I (MAT1512). It deals with the mathematics of change.

Outcomes:

- Calculate and use the derivatives of a function to sketch a graph of the function

- First derivative. Determine the relationship between the rates of change of various quantities in the rates-of-change word problem.

- Solve maximum or minimum word problems using the theory of derivatives.

- Ability to use L’Hopital’s rule to determine limits of indeterminate forms.

- Calculation of the volumes of solids of revolution.

- An improper integral is tested for convergence or divergence and evaluated if convergent.

- Integration techniques to evaluate integrals.

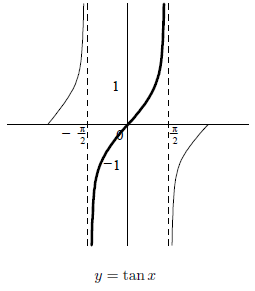
- Taylor polynomial of any order at a given point.

A close up of a map

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**A picture containing hanging, light, boat, traffic

Description automatically generatedA picture containing table, stop, traffic, boat

Description automatically generated**

**A picture containing photo, different, sitting, table

Description automatically generatedA screenshot of a cell phone

Description automatically generatedA picture containing different, photo, hanging, sitting

Description automatically generated**

**Lesson 0**

Trig Revision

Example: NOV 2015 Q2s

Find the exact value of

[1] Compute where the inside function is defined

**wolframalpha**

sin^-1 x

Chart, line chart

Description automatically generated

Let , then

is defined in the 1st and 4th quadrants

[2] Find the value of (Use a reference triangle)

SOH – CAH – TOA

Pythagoras theorem:

y Opposite:

x Adjacent:

2 Hypotenuse:

Pythagoras theorem: SOH, using

Adjacent:

Pythagoras theorem: CAH

If , then

Example: ASS 1 Q4

Find the exact value of

[1] Find out where is defined

is defined in the 1st and 4th quadrants

[2] Find the value of using the reference triangle

SOH – CAH – TOA

Pythagoras Theorem:

y Opposite:

x Adjacent:

2 Hypotenuse:

Let , then

*Use the unit circle, see which in 4th quadrant*

SOH:

If , then:

*Use the unit circle, see where in 1st quadrant*

TOA:

Example:

Find the exact value of

[1] Find where and are defined

is defined in the 1st and 3rd quadrants

is defined in the 1st and 2nd quadrants

[2] Find the value of

Pythagoras theorem:

y Opposite:

x Adjacent:

2 Hypotenuse:

**Let , then**

*Use the unit circle, see which in 3rd quadrant*

TOA:

*Use the unit circle, so at we have (3rd quadrant)*

Radians:

**Let then**

*Use the unit circle, see which in 2nd quadrant*

CAH:

*Use the unit circle, so at we have (2nd quadrant)*

Radians:

*Remember that , so or*

**Lesson 1**

Revision: Limits

Limits are the value a function approaches as the input “approaches” some value. They are used to define continuity, derivatives, and integrals.

We do not care about the output (of a function) at a certain point, but more what happens around the point.

Limits help solve the problem of indeterminate form or

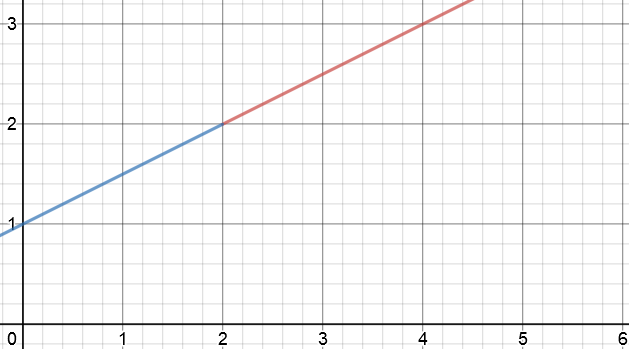
Calculating instantaneous velocity is an example of a limit

Example: The function and the limit differ

*LHS: The limit as x approaches 2 is 2*

*RHS: The limit as x approaches 2 is 2*

*The limit as x approaches 2 is 2*

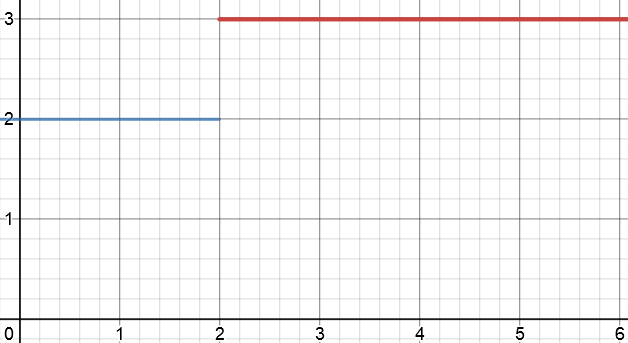


Example: The LHS and RHS limit differ

*LHS: The limit as x approaches 2 is 2*

*RHS: The limit as x approaches 2 is 2*

*The limit as x approaches 2 does not exist*



A close up of a device

Description automatically generatedExample: The limits at inifinity

*LHS: The limit as x approaches -infinity is -1*

*RHS: The limit as x approaches infinity is -1*

*Reciprocal graph. Asymptotes @ x=2, y=-1*

<https://www.youtube.com/watch?v=nJZm-zp639s>

**Common** **(PFGE)** Use these methods in order. If one fails, try the next

[1] Plug in values

*Always start by plugging in the x value*

[2] Algebra. Factorization

*Indeterminate form . Factorize and the plug-in x value*

If you use [2] and you get an answer over zero, then DNE (does not exist)

Other DNE examples

[3] Algebra. Get common Denominator

*Reciprocal Substitute*

[4] Expand Parentheses

*Expand then simplify*

**Uncommon (STA)**

[5] Square root in numerator (in rational expression)

*Multiply by conjugate (differentiation). Remember to change sign of 2nd term*

[6] Trig functions (indeterminate form)

*Special property: or*

*Special property: or*

*because*

[7] Absolute Value

*Piecewise definition of ABS function:*

*Find see if LHS limit = RHS limit*

<https://www.youtube.com/watch?v=nViVR1rImUE>

**Limits at infinity**

*Special property:*

[8] Polynomial/Constant

*Lower degree terms (2x and 5) irrelevant matter here*

[9] Rational

*degree\_N < degree\_D*

*ratio of leading coefficients*

*degree\_N = degree\_D*

*degree\_N > degree\_D*

[10] Trig functions

*Special property: or*

*Special property: or*

*also 0*

[11] Exponential

*eval:*

*eval:*

*Sub*

*Sub*

[11] L’Hopital’s Rule

*Special property:*

Indeterminate forms:

or or or or or or

Example: natural logs

**Continuity**

A function is continuous at a if

*the limit of the function at input a is defined*

Example: MAY 2016 Q4

a)

*Write in the form*

*L’Hopitals Rule*

*Simplify*

b)

**wolframalpha**

lim (1/(xlnx)-x/(x-1)) as x -> 1

*Write in the form*

*Already in the form , therefore L’Hopitals Rule*

TODO

*will reduce to*

a)

*Already in the form , therefore L’Hopitals Rule*

**wolframalpha**

lim (e^(2x)-e^x-x)/(x^2) as x -> 0

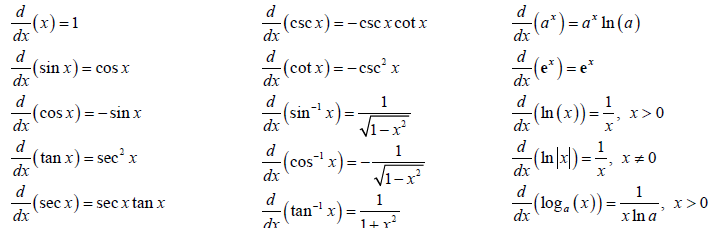
*Remove constant from denominator*

*Again, in the form , therefore L’Hopitals Rule*

**Lesson 2**

Revision: Derivatives P.P.I.C.Q

Derivatives are the slope of a function. It calculates the instantaneous rate of change at each point of the old function.

Common derivatives: 

What is and what is the difference between and ?

is a function that takes one input.

Differentiation incomplete

*differentiation-with-respect-to-x*

or for brevity ,is a function with its input y.

Differentiation complete

*the result of taking the derivative-with-respect-to-x of y*

[1] Power Rule

[2] Product Rule

[3] Quotient Rule

[4] Chain Rule.

Use this when you have a composition function (in the form )

First differentiate Then differentiate

Multiply by

Chain Rule – Exponential

*Common derivative*

Chain Rule – Log

*Common derivative*

Chain Rule – Root

Chain Rule – Chain Rule with the Product Rule

*Product rule -> Chain Rule*

*Chain rule -> Product Rule*

)

[5] Implicit Differentiation

Use this when a function is expressed in terms of both x and y (i.e. a circle of the form

Steps:

Differentiate all terms with respect to

Collect all the on one side

Solve for

Remember: If you are trying to find of a term with , remember to add on a after differentiating (Chain Rule)

e.g.

Find of

*Differentiate all terms with respect to*

Find of

*Differentiate all terms with respect to*

*Product rule*

**Lesson 3**

Integration

This is the antiderivative of a function. It is used for calculating things like areas, volumes, and central points.

**Common integrals**

<http://www.sosmath.com/tables/integral/integ1/integ1.html>

integral

derivative of integral

function

what does all this mean?

integral of the function

infinitesimal displacement along x

limits of integration

[1] Power Rule

[2] Trig

[3] U Substitution

Choose an expression for

Choose an expression for .

*Choose the one that is easiest to differentiate from* . Usually the one in brackets

Integrate and

Substitute original back

Sometimes you might need to manipulate

Sometimes you might need to manipulate

Sometimes you need to re-write the question with brackets

-

-

-

-

-

-

[3] Integration by parts

First try your other methods, in succession

[1] Power Rule [2] Trig [3] U-Substitution

If none of these work, Use integration by parts

Take your original integral, re-write into a form you can use

Choose an expression for

*Choose the one that is easiest to differentiate*

Choose an expression for . (L-I-A-T-E)

Differentiate for

Integrate for

To choose an expression for and more accurately, use LIATE

The first letter that comes up, use for

The second letter that comes up, use for

Logs

Inv trig

Algebraic

Trig

Exponent

In this example:

Algebraic

Exponent

TE are interchangeable

In this example:

Exponent

Trig

Integrate by parts again

We already solved . Therefore:

[3] Trig Substitution

If no other method is working, and you have a radical (), use trig substitution

Sub

*TOA*

Sub

*SOH*

Identify the form you have

*is a constant*

*is*

Compute a set of values

*Substitute*

*Differentiate for*

*solve for (inverse trig)*

Complete a reference

Triangle

*SOH CAH TOA*

Sub triangle values

Sub

*CAH*

**Inverse trig functions**

**Half angle formulas**

**Double angle formulas**

Remember that is assumed positive, so can be written

Apply sum rule

U-Substitution

Sub ref triangle values

**Lesson 4**

Sign Patterns

Critical point

* When the first derivative is zero or does not exist, we have a critical point

Local Maximum

* When the first derivative changes from a positive to a negative, we have a local maximum
* Has a critical point

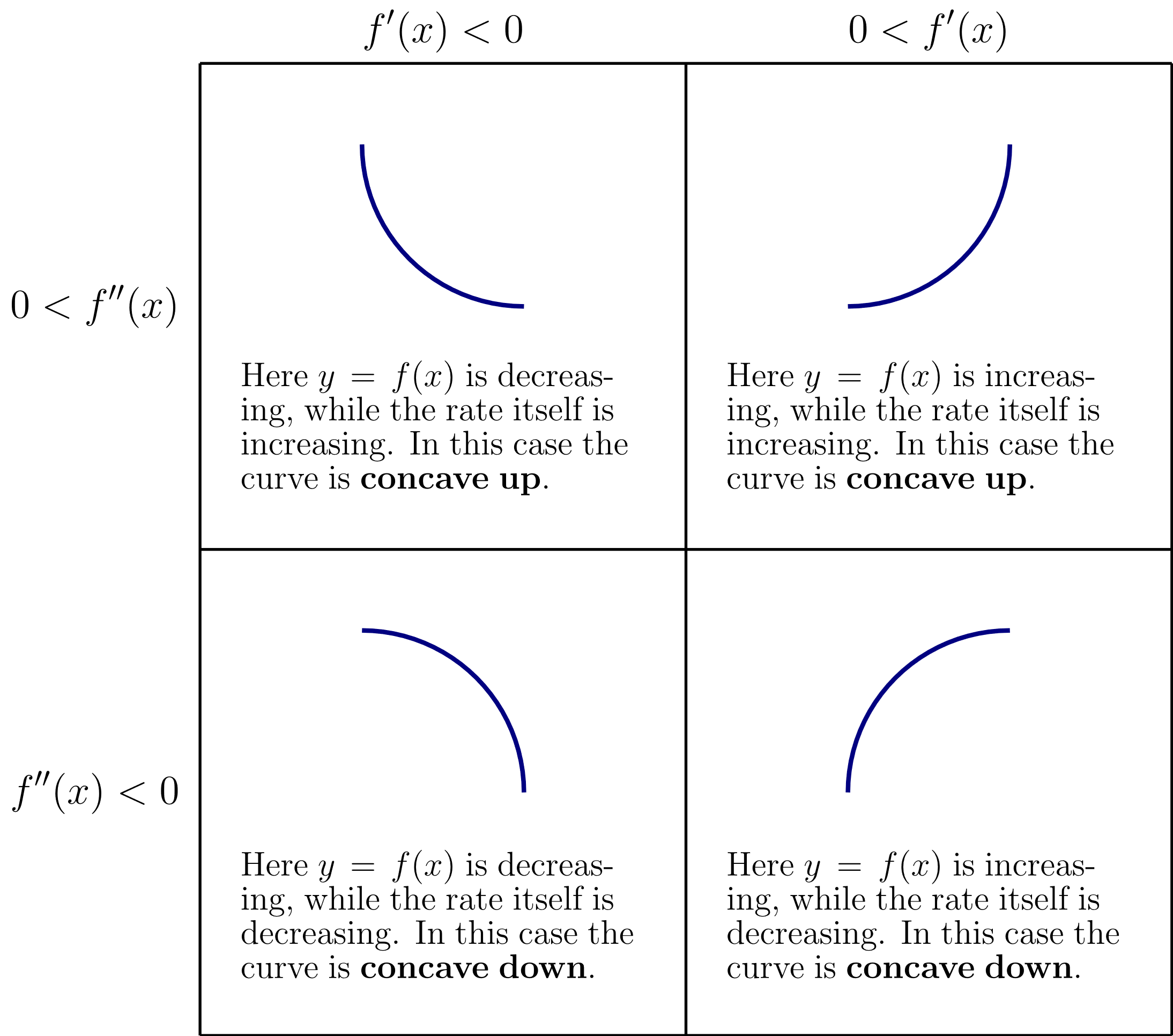
Local Minimum

* When the first derivative changes from a positive to a negative, we have a local maximum
* Has a critical point

Concavity

* Inflection points occur where concavity changes

A close up of a logo

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How to answer sign patterns:

Continuous?

x-intercept:

y-intercept:

Asymptote Horizontal: highest power of x

Asymptote Vertical: limits where undefined

*Undefined where*

Compute

Alternate forms

Sign Patterns

Concavity

Local extrema

Compute

Alternate forms

Sign Patterns (where function is undefined)

*Consider any function restrictions*

Concavity

Local extrema

Continuous?

Positive CONCAVE UP

Negative CONCAVE DOWN

Inflection points at

*Consider any function restrictions*

Example: MAY 2016 Q1

Given the function , where , compute the following:

[1] Compute intercepts

Chart

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**wolframalpha**

intercepts (1-2x^2)/(x^2-1)

[2] Compute horizontal/vertical asymptotes

**wolframalpha**

asymptotes (1-2x^2)/(x^2-1)

horizontal:

vertical:

*Function undefined at*

A close up of a map

Description automatically generated

OR

Horizontal:divide by highest power of x

Vertical: Check limits where undefined

Therefore, vertical asymptotes are

[3] Compute

**wolframalpha**

derivative (1-2x^2)/(x^2-1)

*Function is in the form*

**wolframalpha**

derivative (2x)/(x^2-1)^2

Asymptotes

Horizontal:

Vertical: and

Undefined at and

Sign Pattern

|  |  |  |  |  |  |  |
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Falls

Rises

Example: ASS 1 Q1

1. Compute

**wolframalpha**

derivative (1-2x^2)/(x^2-1)

Write function in the form OR

**wolframalpha**

alternate forms 2x-8/x

Sign Pattern

*Remember that 0 is positive (i.e. where x = 2)*

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
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Falls

Rises

1. Local extreme points

Only happens where (or )

1. Compute

**wolframalpha**

derivative 2x-8/x

Write function in the form OR

**wolframalpha**

alternate forms 2+8/x^2

Example: NOV 2015 Q1

Given the function , compute the following:

**x-intercept(s):**  and

**y-intercept(s):**

**Horizontal Asymptote(s):**

**Vertical Asymptote(s):** ,

*Undefined where and*

**Compute**

**Sign Patterns**

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
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*Add 0 between your undefined points*

**Concavity**

Falls

Rises

Local Extrema

???

**Lesson 5**

Optimization

2

In order to optimize in calculus, we need to:

1. Get the function
2. Minimize the function (Calculate local minimum)
3. Test the function (Test for concavity)

ASS 1 Q2: You are designing a poster to contain 50 of printing with margins of 4 each at the top and bottom and 2 at each side. What overall dimensions will minimize the amount of paper used?

1. Function

Let be the length of the poster

Let be the width of the poster

2. Minimize the function

Product Rule

**Quotient Rule**

Critical point

3. Test the function

**Quotient Rule**

Concavity

or

is the absolute minimum

Example: MAY 2020 Q2

**Function**

Let the length be ,

Let the width be

Thus, the area be the function ,

where is the length and is the width

*Make y the subject*

**[1] Let the area function**

**[2] Minimize the area function**

**wolframalpha**

minimize 400/x+ 4x +82

*You might be tempted to use quotient rule, convert the fraction*

*Minimum where or*

Using the area:

Therefore, the poster has dimensions

**[3] Optional: Test the minimum**

*You might be tempted to use quotient rule, convert the fraction*

**Lesson 6**

Hyperbola

A hyperbolic function is of the form . where and are constants.

shifts the function upwards or downwards

So then hyperbola are two curves that are like infinite bows.

A picture containing diagram

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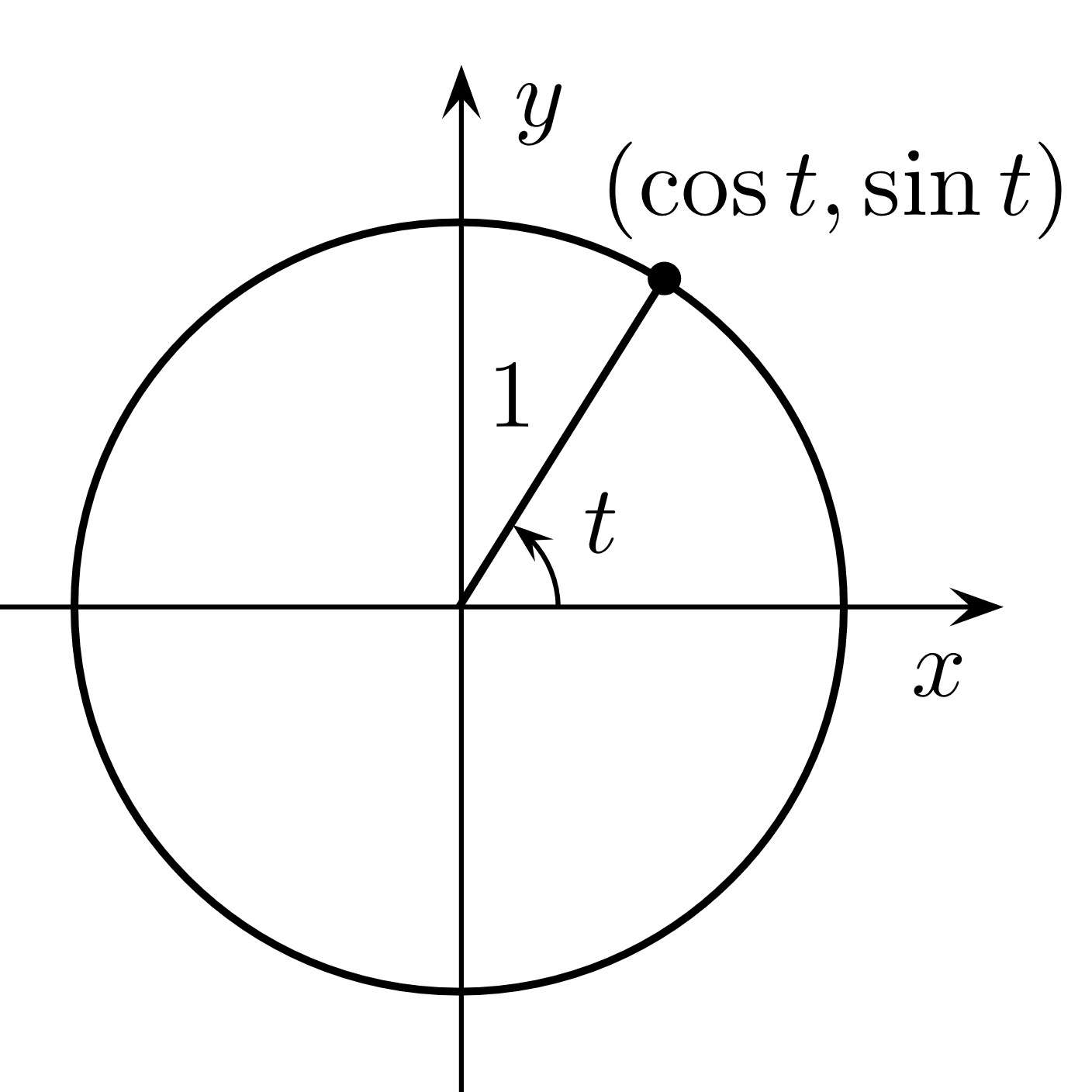
A picture containing sitting, plane, airplane, table

Description automatically generatedA hyperbola is actually the cross section of a cone (refer to MAT2615, quadric surfaces) where a cone is of the form

**Lesson 7**

Hyperbolic Functions

<https://elliptigon.com/hyperbolic-functions-explained/>



Up to this point, we have thought about trig functions in terms of the unit circle of the form .

Coordinates of a general point could be represented as and , where is the parameter (refer to MAT2615, parametric functions)

Diagram

Description automatically generatedEuler’s formula, , gives us a neat, perhaps unexpected, relationship between the trigonometric functions and the exponential function. If we rearrange the terms a little, have the following:

*is the imaginary unit*

If we re-write the above with the imaginary units removed:

*pronounced “*kosh”

pronounced “sinch” or “shine”

In summary, hyperbolic functions have similar names to trig functions, but are defined in terms of the exponential function .

<http://www.mathcentre.ac.uk/resources/workbooks/mathcentre/hyperbolicfunctions.pdf>

Summary:

Diagram

Description automatically generated

Diagram

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Some hyperbolic functions:

Diagram, engineering drawing

Description automatically generatedDiagram

Description automatically generated

Diagram

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**Hyperbolic Identities**

**Other Hyperbolic Functions**

**Hyperbolic Functions**

**Hyperbolic Integrals**

**Hyperbolic Derivatives**

**Lesson 8**

Solids of revolution

<https://www.math24.net/volume-solid-of-revolution-disks-washers/>

If a region in the plane is revolved about a line in the same plane, the resulting object is known as a solid of revolution. Normally, to calculate the area under a curve we would use rectangles (in the limit of infinitely thin rectangles). To calculate the volume of such a shape, we would use disks (in the limit of infinitely thin disks).

<https://www.youtube.com/watch?v=QLHJl2_aM5Q>

A picture containing clock, object, sign

Description automatically generatedA picture containing clock, umbrella

Description automatically generated

Formula for ellipse :

Formula for ellipsoid :

Cross section:

Sphere:

*But remember that the ellipse can be split into halves*

*Integrate . Also multiply by 2*

*between and*

*Evaluate at and at*

: *this is the formula for the volume of a sphere*

:

**Wolfram Alpha**

rotate y=2x, 0<x<3 about the y-axis

**Disk Method**

Example

Let be the closed region between the curve and the origin

Find the volume of the solid obtained by revolving about the axis.

A picture containing logo

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[1] Area

Outer Radius (cross sections)

[2] Volume (Add up cross sections using integration)

*between and*

*Evaluate at the limits and at*

1:

0:

**Washer Method**

Example

Let be the closed region between the curve and

Find the volume of the solid obtained by revolving about the axis.

**Diagram, polygon

Description automatically generated**

[1] Area

Outer Radius

Inner Radius

Area of washer:

[2] Volume (Add up cross sections using integration)

*between and*

*Evaluate at the limits and at*

1:

0:

**Washer Method**

Example: MAY 2016 Q5

Let be the closed region between the curve and

Find the volume of the solid obtained by revolving about the axis.

**Wolfram Alpha**

y=2x^2+1 and y=2x+1

**Chart, line chart

Description automatically generated**

*ROTATE AROUND Y AXIS*

*Write function in terms of x*

[1] Area

Outer Radius

*keep fractions!*

s

Inner Radius

*keep fractions!*

Intersection points

Let

or

Area of washer: *Y AXIS*

[2] Volume (Add up cross sections using integration)

*between and*

*Evaluate at the limits and at*

1:

3:

But volume can only be positive, therefore

**Wolfram Alpha**

revolve region between y=2x^2+1 and y=2x+1, about the y-axis

Let be the closed region between the curve and

Find the volume of the solid obtained by revolving about the line

This is still a rotation around the, just shifted up one unit

**Wolfram Alpha**

y=x^2 and y=2x

y=x^2-1 and y=2x-1 shifted upwards

**Chart, line chart

Description automatically generated**

[1] Area

Outer Radius

Inner Radius

Area of washer:

Intersection points

Let

or

[2] Volume (Add up cross sections using integration)

between and

between and

:

:

TODO

**Lesson 9**

Taylor series

From MAT2615

Taylor Polynomials functions

This helps us obtain accurate approximations can be obtained by using higher order derivatives.

Example: Find the 4th degree Taylor polynomial for centered at and use it to approximate

*General formula for Taylor polynomial*

*Substitute into general formula*

*Therefore, approximation of*

**Wolfram Alpha**

series f(x)=2sqrt(x) at a = 4 order 4

**Symbolab**

Example: MAY 2016 Q3

Example: MAY 2015 Q3

Study List

[x] Sign patterns

[x] Optimization

[ ] Rolle’s Theorem

[ ] Mean Value Theorem

[x] Area between curves, solids of revolution

[x] Integration

[x] Integration U Substitution

[x] Integration by parts

[x] Integration by trig substitution

[x] Hyperbolic functions

[x] Solids of revolution

[x] Taylor polynomials

Revision List

[x] Trig identities/Formulas

[x] Derivatives – Extrema

[x] Derivatives – Integrals

[ ] Derivatives – Common Integrals